Stochastic Growth in One Dimension and Gaussian Multi-Matrix Models

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Abstract

We discuss the space-time determinantal random field which arises for the PNG model in one dimension and resembles the one for Dyson's Brownian motion. The information of interest for growth processes is carried by the edge statistics of the random field and therefore their universal scaling is related to the edge properties of Gaussian multi-matrix models.

1 Introduction

There is a huge variety of stochastic growth processes and we refer to the recent monographs by Barabási, Stanley[1] and Meakin[2]. Amongst them the KPZ (Kardar, Parisi, Zhang) growth models enjoy a particular popularity in theoretical circles. Prototypes are the Eden growth, where each perimeter site of the current cluster is filled after an exponentially distributed waiting time, and ballistic deposition, where there is a low intensity random flux of incoming particles which then attach to the current surface profile. Thus in broad terms KPZ growth is characterised by being stochastic with a

- local growth rule,
- smoothening mechanism.

The list of experiments well described in terms of KPZ is rather short, see Mylls *et al.*[3] for recent experiments on slow combustion fronts. But one obvious reason for its popularity is that KPZ growth is rather close to the stochastic dynamics commonly studied in Statistical Mechanics with the fine twist that it does not satisfy detailed balance (KPZ growth is not stochastically reversible). There is a second reason, however: through a Cole-Hopf type transformation KPZ growth maps to directed first passage percolation. Thereby techniques and insights from the theory of disordered systems come into play. In fact, there are some rather close analogies, one example being the rigorous discussion of the overlap for directed polymers[4].

Just before the previous ICMP congress it came as a total surprise that KPZ growth in one spatial dimension (the surface is the graph of a function over $\mathbb{R}$) is linked to Gaussian random matrices. For example, the height, $h(\tau)$, at time τ above a given reference point grows at constant speed v_0 with some random fluctuations,

$$h(\tau) = v_0\tau + \tau^{1/3}\xi \tag{1}$$

for large τ. If at the reference point the macroscopic profile has a non-zero curvature, then ξ has the same distribution function as the largest eigenvalue of a GUE random matrix, which in the community is known as the Tracy-Widom distribution function F_2, $\mathbb{P}(\xi \leq x) = F_2(x)$. F_2 is related to the Hastings-McLeod solution of the Painlevé II differential equation. The scaling exponent $1/3$ and the random amplitude ξ in (1) are expected to be valid for all one-dimensional KPZ growth models.

The purpose of our contribution is to explain how this connection arises. The rough indication can be given already now: For some very particular growth models in the KPZ class, there is a determinantal process in the background. Its structure has some similarity to the determinantal process appearing in the context of Gaussian multi-matrix models. Since the relevant information is linked to edge scaling, in the scaling limit the GUE edge distribution arises.

2 GUE, Dyson's Brownian motion, and the Airy process

Dyson[5] considers an Ornstein-Uhlenbeck process, $A(t)$, $t \in \mathbb{R}$, on the space of $N \times N$ Hermitian matrices. Its transition probability is given through a

Mehler type formula as

$$\mathbb{P}(A(t) \in dA' | A(0) = A) = \frac{1}{Z} \exp[-\frac{1}{N} \mathrm{Tr}(A' - qA)^2/(1-q^2)] dA' \quad (2)$$

with $q = e^{-t}$. In particular, for $t \to \infty$ the process converges to the GUE ensemble given by

$$Z^{-1} \exp[-\frac{1}{N} \mathrm{Tr} A^2] dA \,. \quad (3)$$

For the joint distribution of the stationary process at times ordered as $t_0 < t_1 < ... < t_m$ one obtains therefore

$$\begin{aligned} &\mathbb{P}(\{A(t_j) \in dA_j, j = 0, 1, ..., m\}) \\ &= \frac{1}{Z} \exp\Big[-\frac{1}{N} \Big(\sum_{j=1}^{m} \mathrm{Tr}(A_j - q_j A_{j-1})^2/(1-q_j^2) + \mathrm{Tr} A_0^2 \Big) \Big] \prod_{j=0}^{m} dA_j \quad (4) \end{aligned}$$

with $q_j = \exp[-(t_j - t_{j-1})]$. (4) is the Gaussian multi-matrix model of the title.

The process $A(t)$ induces a process on the eigenvalues $\lambda_N(t) \leq ... \leq \lambda_1(t)$ of $A(t)$. It is stationary by construction and happens to be again Markov. In fact, it satisfies the set of stochastic differential equations

$$d\lambda_j(t) = \Big(-\frac{1}{N} \lambda_j(t) + (\beta/2) \sum_{i=1, i \neq j}^{N} \big(\lambda_j(t) - \lambda_i(t) \big)^{-1} \Big) dt + db_j(t) \,, \quad j = 1, ..., N \,, \quad (5)$$

with $\{b_j(t),\, j = 1, ..., N\}$ a collection of N independent standard Brownian motions. In case of Hermitian random matrices $\beta = 2$. The repulsive drift ensures that $\lambda_{j+1}(t) < \lambda_j(t)$ for all t, any j. Clearly (5) has the unique stationary distribution

$$\frac{1}{Z} \exp\Big[-\frac{1}{N} \sum_{j=1}^{N} \lambda_j^2 \Big] \prod_{1 \leq i < j \leq N} |\lambda_i - \lambda_j|^\beta \prod_{j=1}^{N} d\lambda_j \,. \quad (6)$$

There is a way to rewrite (5) which turns out to be computationally very powerful. We define a random field ϕ_N over $\mathbb{R}^2$ through

$$\phi_N(x, t) = \sum_{j=1}^{N} \delta(x - \lambda_j(t)) \,. \quad (7)$$

Then ϕ_N is determinantal in the sense that its moments (correlation functions) have a determinantal structure of the form

$$\langle\prod_{j=1}^{m}\phi_N(x_j,t_j)\rangle = \det\{R_N(x_i,t_i;x_j,t_j)\}_{i,j=1}^{m} \tag{8}$$

for distinct times $t_1,\dots,t_m$. The defining kernel R_N can be written in terms of the harmonic oscillator Hamiltonian

$$H_N = -\frac{1}{2}\partial_x^2 + \frac{1}{2N^2}x^2\,, \tag{9}$$

which has eigenvalues $E_n = n/N$, $n = 1, 2, \dots$. Let K_N be the Hermite kernel which is the spectral projection onto $\{H_N \le 1\}$. Then

$$R_N(x,t;x',t') = \Big(e^{-tH_N}\big(K_N - 1\!\!1\Theta(t-t')\big)e^{t'H_N}\Big)(x,x') \tag{10}$$

with $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t \le 0$. In particular, at $t = 0$ (or any other time by stationarity)

$$\langle\prod_{j=1}^{m}\phi(x_j,0)\rangle = \det\{K_N(x_i,x_j)\}_{i,j=1}^{m}\,, \tag{11}$$

as well known from the random matrix bible by Mehta[6].

A natural question is to study the statistics of lines close to the, say upper, edge. In our units the top line fluctuates at level $\sqrt{2}N$, as can be seen by equating the Fermi energy E_{F}, $E_{\mathrm{F}} = 1$, with the energy of the confining potential of (9). Thus we shift our attention to $x = \sqrt{2}N$ and linearise there the potential, a procedure which should be accurate for large N. Then the imaginary time Schrödinger equation for (9) goes over to

$$\partial_t\psi = (-\frac{1}{2}\partial_x^2 + \frac{1}{N}\sqrt{2}x)\psi\,. \tag{12}$$

It becomes N-independent under the scaling $t \rightsquigarrow N^{2/3}t$, $x \rightsquigarrow N^{1/3}x/\sqrt{2}$, resulting in the Schrödinger equation with Airy Hamiltonian

$$\partial_t\psi = H\psi\,,\quad H = -\partial_x^2 + x\,. \tag{13}$$

We have identified the edge scaling and conclude that, in the sense of convergence of moments,

$$\lim_{N\to\infty} \frac{1}{\sqrt{2}} N^{1/3} \phi_N(\sqrt{2}N + \frac{1}{\sqrt{2}} N^{1/3} x\,,\ N^{2/3} t) = \phi(x,t)\,. \tag{14}$$

The prefactor comes from the spatial volume element when integrating both sides in (14) over a compactly supported test function. ϕ is called the Airy random field. Since ϕ_N is determinantal, so must be its limit. Hence, for distinct times $t_1, ..., t_m$,

$$\langle \prod_{j=1}^{m} \phi(x_j, t_j) \rangle = \det\{R(x_i, t_i; x_j, t_j)\}_{i,j=1}^{m}\,, \tag{15}$$

compare with (8). The defining kernel is now given through

$$\begin{aligned} R(x,t;x',t') &= \Big(e^{-tH}\big(K - 1\!\!1\Theta(t-t')\big)e^{t'H}\Big)(x,x') \qquad (16)\\ &= \operatorname{sign}(t'-t)\int d\lambda \Theta(\lambda(t-t'))e^{\lambda(t'-t)}\mathrm{Ai}(x-\lambda)\mathrm{Ai}(x'-\lambda)\,, \end{aligned}$$

where $\operatorname{sign}(t) = 1$ for $t \geq 0$ and $\operatorname{sign}(t) = -1$ for $t < 0$, Ai the Airy function, and K the spectral projection onto $\{H \leq 0\}$. K is known as Airy kernel. For fixed t the limit is studied by Forrester[7], and Tracy, Widom[8]. Some aspects of the multi-matrix model are discussed by Eynard[9].

The top line of the Airy random field is called the Airy process[10], denoted by $\mathcal{A}(t)$. Its joint distributions can be written in a concise way. Let $t_1 < ... < t_m$ and let us define the operator, $R^{(m)}$, on $L^2(\mathbb{R} \times \{1, ..., m\}, dx)$ through the integral kernel $R(x, t_i; x', t_j)\chi(\{x > \xi_i\})\chi(\{x' > \xi_j\})$, $i, j = 1, ..., m$. Then $R^{(m)}$ is trace class and

$$\begin{aligned} \mathbb{P}(\{\mathcal{A}(t_1) \leq \xi_1, ..., \mathcal{A}(t_m) \leq \xi_m\}) &= \mathbb{P}(\{\phi(x,t_j) = 0 \quad \text{for} \quad x > \xi_j\,,\ j = 1, ..., m\})\\ &= \det(1 - R^{(m)})\,. \qquad (17) \end{aligned}$$

The Airy process has continuous sample paths and is stationary, by construction. Some more explicit expressions for joint distributions are given in [11, 12, 13, 14]. In particular, $\langle \mathcal{A}(0)^2 \rangle = 0.81325...$, $\langle (\mathcal{A}(0) - \mathcal{A}(t))^2 \rangle = 2|t|$ for small t, and $\langle \mathcal{A}(0)\mathcal{A}(t) \rangle - \langle \mathcal{A}(0) \rangle^2 = t^{-2} + \mathcal{O}(t^{-4})$ for large t.

3 Polynuclear growth, its determinantal random field, and edge scaling

For the polynuclear growth (PNG) model the height function at time τ takes integer values, $h(x,\tau) \in \mathbb{Z}$, $x \in \mathbb{R}$, $\tau \geq 0$. $x \mapsto h(x,\tau)$ has jumps of unit size only. Upward steps move, in the x direction, with velocity -1 and downward steps with velocity 1. They annihilate each other upon collision, which is the required smoothening mechanism. The surface grows through nucleation. At such an event, say (x_1, τ_1), the height $h(x, \tau_1)$ is increased by one unit at x_1, thereby creating an upward and downward step, which move apart immediately under the deterministic part of the dynamics. The nucleation events are Poisson in space-time with intensity 2. We consider the droplet geometry which is enforced by allowing nucleations only in the space interval $[-\tau, \tau]$. Initially $h(x, 0) = 0$ and, by assumption, $h(x, \tau) = 0$ for $|x| > \tau$. Obviously, according to our criteria of the Introduction, the PNG model is in the KPZ class.

For PNG, as just described, there is no determinantal process in sight. The miracle happens through the RSK (Robinson, Schensted, Knuth) construction. The idea is to extend the model with additional lines which record the information lost in annihilation events. Thus we introduce the lines $h_j(x,\tau)$, $j = 0, -1, -2, \ldots$, and set $h_0(x,\tau) = h(x,\tau)$. Initially $h_j(x, 0) = j$. h_0 evolves according to the PNG specified above. Given $h_0(x,\tau)$ all lower lying lines have a deterministic dynamics. The steps of line h_j, $j \leq -1$, move with velocity ± 1, as before. A nucleation, say at space-time point (x,τ), takes place whenever in line $j+1$ an upward and downward step annihilate each other at (x,τ). Thus $x \mapsto h_j(x,\tau)$ has jumps of unit size only, $h_j(x,\tau) = j$ for $|x| > \tau$, and $h_j(x,\tau) > h_{j-1}(x,\tau)$. Furthermore there is a random index j_0 such that, for $j \leq j_0$, $h_j(x,\tau) = j$ for all x. Let us set, at fixed time τ,

$$\eta_\tau(j,t) = \begin{cases} 1 & \text{if there is a height line passing through}\,(j,t)\,, \\ 0 & \text{if there is no such line}\,, \end{cases} \tag{18}$$

with $j \in \mathbb{Z}$, $t \in [-\tau, \tau]$. $\eta_\tau(j,t)$ is a determinantal random field over $\mathbb{Z} \times [-\tau, \tau]$. Its top line is the object of interest, since it coincides with PNG.

To write down the defining kernel for η_τ we set, as operators on $l_2 = l_2(\mathbb{Z})$,

$$H_{\mathrm{d}}\psi(j) = -\psi(j-1) - \psi(j+1)\,, \tag{19}$$

$$H_\tau\psi(j) = -\psi(j-1) - \psi(j+1) + \frac{j}{\tau}\psi(j)\,, \tag{20}$$

and B_τ the spectral projection on $\{H_\tau \le 0\}$. B_τ is known as discrete Bessel kernel. Then

$$R_\tau(j,t;j',t') = \Big(e^{-tH_{\mathrm{d}}}\big(B_\tau - 1\!\!1\Theta(t-t')\big)e^{t'H_{\mathrm{d}}}\Big)_{jj'}\,. \tag{21}$$

The moments of $\eta_\tau(j,t)$ are given by the formula analogous to (8). For large τ, $h_0(x,\tau) \cong 2\sqrt{\tau^2 - x^2}$, $|x| < \tau$. Thus $\eta_\tau(x,t)$ is not stationary in t, which is reflected in (21) by the fact that $H_{\mathrm{d}} \neq H_\tau$.

The correct edge scaling can be guessed as for Dyson's Brownian motion, where in spirit τ is equated with N. We focus our attention on a space-time window of width $\tau^{2/3}$ and height $\tau^{1/3}$ centred at $t = 0$ and $j = 2\tau$. Properly rescaled, compare with (12), H_τ becomes

$$H_\tau\psi(x) = \tau^{2/3}\big(-\psi(x-\tau^{-1/3}) - \psi(x+\tau^{-1/3}) + 2\psi(x) + \tau^{1/3}(x/\tau)\psi(x)\big)\,, \tag{22}$$

which converges to the Airy operator H as $\tau \to \infty$. This argument overlooks that even in rescaled coordinates the lines have still a systematic curvature. The correct limit is thus, $[\cdot]$ denoting integer part,

$$\lim_{\tau\to\infty} \tau^{1/3}\eta_\tau([2\tau + \tau^{1/3}(x - t^2)],\, \tau^{2/3}t) = \phi(x,t) \tag{23}$$

with ϕ the Airy random field. In particular, the top line converges to the Airy process $\mathcal{A}(t)$.

We summarise our discussion as

Theorem 3.1. *Let $h(x,\tau)$ be the PNG model in the droplet geometry. Then, in the sense of weak convergence of finite-dimensional distributions,*

$$\lim_{\tau\to\infty} \tau^{-1/3}\big(h(t\tau^{2/3}\,,\,\tau) - 2\tau\big) = \mathcal{A}(t) - t^2\,. \tag{24}$$

Proof. For $t = 0$ this is the celebrated result of Baik, Deift, Johansson[15] on the length of the longest increasing subsequence of a random permutation. They use orthogonal polynomials and Riemann-Hilbert techniques in their asymptotic analysis. Johansson[16] develops an approach through Fredholm determinants, which is also the basis of the proof by Prähofer, Spohn[10] for the time-extended case. □

Theorem 3.1 is stated for the reference point $x = 0$. An analogous limit holds for any other reference point $x = a\tau$, $|a| < 1$.

For a space-time discrete version of the PNG droplet Johansson[17] proves the analogue of Theorem 3.1 including tightness. While the details are long, it is likely that Theorem 3.1 can be strengthed to weak convergence of path measures. As established by Johansson[18], for the Aztec diamond the border between the frozen and disordered zones is also governed by the Airy process.

4 Extensions

What is so special about the droplet? From the point of view of growth processes other initial conditions look more natural, e.g. the initially flat surface $h(x, 0) = 0$. The RSK dynamics can be implemented for any choice of initial data and set of nucleation events. However the determinantal property of the resulting line ensemble is fragile.

We illustrate our point by three examples.

(i) *half droplet*. In a discrete space-time setting this problem is studied recently by Sasamoto and Imamura[19]. The rule is to simply suppress all nucleation events for $x < 0$. In addition there is a source at $x = 0$ with rate γ. For $0 \leq \gamma < 1$ the additional mass is incorporated in the bulk with no change in the macroscopic profile. The height fluctuations at $x = 0$ satisfy (1) with F_2 replaced by F_4, i.e. the distribution function of the largest eigenvalue of a GSE random matrix in the limit of large N [20]. $\gamma = 1$ is critical and F_2 in (1) is replaced by F_1 [20], i.e. the distribution function of the largest eigenvalue of a GOE random matrix[21]. For $\gamma > 1$ the macroscopic profile develops a linear portion, starting at $x = 0$, at height $(\gamma + \gamma^{-1})\tau$ and joining tangentially the profile for $\gamma = 0$. The height fluctuations are then of order $\sqrt{\tau}$ and Gaussian. The RSK dynamics results in a line ensemble, which for fixed τ has the following weight. At $x = \tau$ the boundary condition is $h_j(\tau, \tau) = j$. The lines have jump size one and are not allowed to cross. Under these constraints the jump points are uniformly Lebesgue distributed. If this construction would be extended to $x = -\tau$ with the boundary condition $h_j(-\tau, \tau) = j$, the resulting line ensemble is the one of the PNG droplet. For the half-droplet the lines end at $x = 0$ and there is an extra weight from

the source at the origin, which reads

$$\exp[(\log\gamma)\sum_{j=-\infty}^{0}\big(h_{2j}(0,\tau)-h_{2j-1}(0,\tau)-1\big)]\,. \tag{25}$$

For $\gamma=1$ the extra weight (25) equals 1 and the weight for the line ensemble can be written as a product of determinants. However, the correlation functions are not determinantal because of the free boundary at $x=0$. By an identity of de Bruijn, gap probabilities are square roots of determinants, thus yielding GOE edge statistics at $x=0$ in the scaling limit. For $\gamma=0$ the lines at $x=0$ are constrained as $h_{2j}(0,\tau)-h_{2j-1}(0,\tau)=1$, which resembles Kramers degeneracy, thus GSE. An asymptotic analysis is available only for $\gamma=0,1$ [19] with the result that $\tau^{2/3}$ close to the origin the height statistics is governed by the largest eigenvalue of the transition ensemble governing the crossover from GOE, resp. GSE, at $x=0$ to GUE in the bulk. For $\gamma>1$, at $x=0$ the top line separates the distance $(\gamma+\gamma^{-1}-2)\tau$ from $h_{-1}(0,\tau)$ which remains located roughly at 2τ, as for $\gamma\le 1$. This explains the Gaussian fluctuations for $\gamma>1$.

(ii) *stationary PNG.* If initially the upward steps are Poisson distributed with density ρ_+ and the downward steps with density ρ_-, then PNG on the whole real line is a space-time stationary growth process, provided $\rho_+\rho_-=1$. To determine $h(0,\tau)$ it suffices to know the nucleation events in the backward light cone $\{(x,t)|0\le t\le\tau,\,|x|\le\tau-t\}$. In fact it also suffices to only know the nucleation events in the forward light cone of the origin, $\{(x,t)|0\le t\le\tau,\,|x|\le\tau\}$. The reason is that for stationary PNG model the intersection points of the diagonals $\{x=\pm t\}$ with the world lines for steps are again Poisson distributed. Thus we arrive at a set-up rather similar to the PNG droplet. The additional feature is that there are sources of nucleation events at the boundaries $\pm\tau$. The sources are Poisson in time with rates α_+, resp. α_-, $\alpha_+\ge 0$, $\alpha_-\ge 0$, stationarity being equivalent to $\alpha_+\alpha_-=1$. As before, a line ensemble is generated through the RSK dynamics. At time τ, the boundary conditions $h_j(\pm\tau,\tau)=j$, $j=-1,-2,...$, hold. However, $h_0(\pm\tau,\tau)$ can now take arbitrary positive integer values. They have geometric weight, i.e. the weight

$$\exp[(\log\alpha_\pm)h_0(\pm\tau,\tau)]\,. \tag{26}$$

The step points are Lebesgue distributed, constrained by non-crossing and the boundary conditions at $\pm\tau$. Let Ω be the collection of all such lines.

By running RSK backwards in time, one will end up with $h_j(0,\tau) = j$ and $h_0(0,\tau) = n$ for $\tau > 0$ but sufficiently small. Ω decomposes accordingly as $\Omega = \cup_{n\geq 0}\Omega_n$. Ω_n has the total weight $(\alpha_+\alpha_-)^n Z_0$, $Z_0 = \exp[t(\alpha_+ - \alpha_+^{-1}) + t(\alpha_- - \alpha_-^{-1})]$. Stationary growth corresponds to the sector Ω_0 carrying weight $\mathbb{W}_0$ and probability $\mathbb{P}_0 = \mathbb{W}_0/Z_0$ for the lines. However, only the line ensemble with weight $\mathbb{W}$ on Ω is determinantal. Fortunately, one has the simple relationship

$$\begin{aligned} \mathbb{P}_0(\{h_0(0,\tau) = k\}) &= Z_0^{-1}\mathbb{W}_0(\{h_0(0,\tau) = k\}) \\ &= Z_0^{-1}\big(\mathbb{W}(\{h_0(0,\tau) = k\}) - \alpha_+\alpha_-\mathbb{W}(\{h_0(0,\tau) = k-1\})\big) . \end{aligned} \tag{27}$$

Stationary PNG is described by the spatial derivative of a determinantal process. (27) serves as a starting point for an asymptotic analysis, see Baik, Rains[22] and Prähofer, Spohn[23].

(iii) *flat initial conditions*. One sets $h(x,0) = 0$ and allows for nucleation events on the entire real line. For the RSK construction it is necessary to first restrict to the periodic box $[-\ell,\ell]$. Then $h_j(-\ell,\tau) = h_j(\ell,\tau)$. As before, the lines are constrained on non-crossing and unit jumps. In addition they have to satisfy

$$\min_{|x|\leq\ell}\big(h_j(x,\tau) - h_{j-1}(x,\tau)\big) = 1 . \tag{28}$$

An asymptotic analysis of the line ensemble with these constraints is not available. By completely different methods Baik, Rains[20] prove that, first taking $\ell \to \infty$, the height fluctuations at $x = 0$ are distributed as the largest GOE eigenvalue, i.e. F_1 replaces F_2 in (1). On this basis, a possible guess for the full height statistics, in the limit $\tau \to \infty$, is Dyson's Brownian motion at $\beta = 1$. We will have to wait for the next ICMP congress for a confirmation (or a counter argument).

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